# Isotropic Cosmological Singularities II. The Einstein-Vlasov system.

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**Abstract**

We consider the conformal Einstein equations for massless collisionless gas cosmologies which admit an isotropic singularity. After developing the general theory, we restrict to spatially-homogeneous cosmologies. We show that the Cauchy problem for these equations is well-posed with data consisting of the limiting particle distribution function at the singularity.

## 1 Introduction

In the accompanying paper, (Anguige and Tod 1998, hereafter ATI), we considered the Cauchy problem for the conformal Einstein equations with a perfect fluid source and with data given at an isotropic singularity. Among other results, we were able to show that a perfect fluid cosmology, with a polytropic equation of state, for which the Weyl tensor vanished at an initial, isotropic singularity was conformally-flat: if the Weyl tensor is zero initially then it is always zero. This is a strong result, and one naturally wonders how robust it is. In particular, does it continue to hold with other matter models?

In this, the second paper of the series, we consider the same Cauchy problem but with a source given by a different matter model, a massless collisionless gas. This is the Einstein-Vlasov system. Our long-term aim is to prove existence and uniqueness for this system with data at an isotropic singularity. However, in the present paper we only partially achieve this aim: we are able to prove what we want only with the restriction of spatial homogeneity. This is sufficient to investigate the Weyl tensor problem de-

scribed above, but of course the long-term aim remains.

In the Einstein-Vlasov system (see e.g. Ehlers 1971, Rendall 1992, 1997) the matter is determined by the distribution function $f(x^a, p_b)$, a positive function on the cotangent bundle of space-time, which is interpreted as the density of particles at the space-time point labelled by $x^a$ which have 4-momentum $p_b$. The collisionless assumption is translated as the assumption that the distribution function is constant along the geodesic flow, giving an equation known in this context as the Vlasov (or Liouville) equation. The Einstein equations are written down with a stress-tensor defined as an invariant second moment of $f$ integrated over the fibre of the cotangent bundle. Thus $f$ is constrained by the condition that this moment exist. We will consider only massless matter, which means that $f$ is supported on the null-cone at each point. For analytical convenience, we will assume that $f$ is supported compactly on the cone and away from the vertex of the cone.

The massless assumption will imply that the theory has a good transformation under conformal rescaling of the metric. For example, the distribution function is constant along null geodesics, which are conformally-invariant. Also the stress-tensor is trace-free, which is the case for radiation perfect-fluid where it is known from ATI that the conformal factor can be chosen to be smooth at an isotropic singularity.

We refer to the accompanying paper for the background to conformal rescaling. Recall only that the physical metric $\tilde{g}_{ab}$ of space-time $\tilde{M}$ is related by rescaling to an unphysical metric $g_{ab}$ on a larger space-time $M$ according to:

$$\tilde{g}_{ab} = \Omega^2 g_{ab}$$

where the conformal factor $\Omega$ vanishes at a smooth surface $\Sigma$ in the extended space-time $M$. As remarked above, the massless assumption will enable us to suppose that $\Omega$ is smooth at $\Sigma$.

After establishing the conformal transformation properties of the matter model, we shall seek Bondi-type expansions of space-time quantities as power-series in the conformal factor, which becomes a good time-coordinate in the unphysical (rescaled) space-time. In this way, we are able to isolate the degrees of freedom of the problem, or equivalently to identify the Cauchy data. The data are very different from the perfect-fluid case. We need the first and second fundamental forms of the singularity surface $\Sigma$

and the initial distribution function $f^0$. These turn out to have very strong constraints among them, so strong in fact that *the initial distribution function $f^0$ uniquely determines the first and second fundamental forms of the singularity surface*. The initial distribution function is subject to a single, integral constraint which can be interpreted as a 'zero drift' condition.

Recall from ATI that in the perfect-fluid case, the second fundamental form of $\Sigma$ is necessarily zero and the first fundamental form is free data. There is no free data for the matter (apart from the equation of state). One can say aphoristically that *the metric determines the matter*. From the previous paragraph, we see that circumstances are quite different in the Einstein-Vlasov case where *the matter determines the metric*.

Because the second fundamental form of the singularity necessarily vanishes for perfect fluid, the condition of initially-vanishing Weyl tensor forces the physical cosmology to be FRW, so that, as we noted above, the Weyl tensor is always zero if it is initially zero (ATI). With Einstein-Vlasov the situation is rather different. The second fundamental form is determined by $f^0$ and its vanishing is an integral constraint on $f^0$, related to the vanishing of a third moment. It is quite possible, as we shall see in section 6.6, that the lower moments of $f^0$ are such that the initial Weyl tensor is zero but there are higher moments which allow the later evolution to diverge from FRW, so that non-zero Weyl tensor emerges.

Once we have identified the data, we turn to the problem of existence and uniqueness of solutions. For this, we restrict to spatially- homogeneous cosmologies. Then the Einstein equations become a system of ordinary differential equations with a singularity in the 'time' and a source involving integrals over the distribution function $f$, while the Liouville equation remains a partial differential equation for $f$. We prove existence and uniqueness for this coupled system by a method based on results of Rendall (1994) and Rendall and Schmidt(1991).

The contents of the paper are as follows. We begin in section 2 with a review of relativistic kinetic theory and then in section 3 consider the problem of conformally-rescaling the massless Einstein-Vlasov (henceforth EV) system. The Bondi expansions are begun in section 4, which includes the proof that the constraint relating $f^0$ and the metric $a_{ij}$ of $\Sigma$ determines $a_{ij}$ uniquely. In section 5, we fix the conformal gauge for the EV system which enables us to simplify the expressions found for the data in section 4

by eliminating some gauge quantities. Section 6 is the heart of the paper. We adapt the expansions of section 4 to a spatially-homogeneous cosmology, identify the data and then prove, in Theorem 6.1, that solutions exist and are unique given suitable data. We consider what data lead to FRW cosmologies and show in particular that there are Bianchi-type V solutions in which the Weyl tensor is zero initially but not for all time.

## 2 Relativistic kinetic theory

In kinetic theory, the matter content of spacetime $\tilde{M}$ is taken to consist of a collection of particles which move on geodesics between collisions. The motion of a freely falling particle is determined by the Lagrangian:

$$L(x^c, \dot{x}^d) = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b \tag{1}$$

since the Euler-Lagrange equations are equivalent to the (affine) geodesic equation. Writing $p_a = g_{ab}\dot{x}^b = \frac{\partial L}{\partial \dot{x}^a}$ = canonical momentum, we get the Hamiltonian

$$H(x^c, p_d) = \frac{1}{2} g^{ab} p_a p_b \tag{2}$$

and thus we obtain canonical equations for the geodesics. $H$ is constant along each particle world-line, $p_a$ is the 4-momentum, and $m = \sqrt{p_a p^a}$ is the mass of the test particle. To take into account all possible particle positions and momenta, we must consider the one-particle phase space $P$ . This is the collection of instantaneous states $(x^a, g_{ab}\dot{x}^b)$ of a single particle and is the future causal part of the spacetime cotangent bundle.

States having a given mass $m \geq 0$ form the seven-dimensional submanifold $P_m$ of $P$. On $P_m$ we take $(x^a, p_i)$ as local coordinates, $p_0$ being determined by the equation $g^{ab}p_a p_b = m^2$, and the requirement that $p_a$ be future directed (conventionally, $a, b, c, .. = 0-3$ while $i, j, k, .. = 1-3$). The free-fall trajectories define on $P$ a congruence of curves, along which $H$ is constant. The geodesic spray $\mathcal{L}$ is a vector field along these curves defined, in local coordinates, by

$$\mathcal{L} = g^{ab} p_a \frac{\partial}{\partial x^b} - \frac{1}{2} p_a p_b \frac{\partial g^{ab}}{\partial x^c} \frac{\partial}{\partial p_c} \tag{3}$$

The cotangent space to $M$ at $x$ is a flat Lorentzian manifold, and on the submanifold $P_m(x)$ there exists an invariant volume measure $\pi_m$ given by

$$\pi_m = \frac{(-g)_x^{-1/2}}{p^0} d^3 p_i. \tag{4}$$

The distribution of particles and momenta is described by a scalar function $f = f(x^a, p_b)$ on $P$, and the central result of relativistic kinetic theory is the equation

$$\mathcal{L}(f) = g \tag{5}$$

where $g$ describes the density of particle collisions. Hence the condition that $f$ represents a collisionless gas is simply

$$\mathcal{L}(f) = 0 \tag{6}$$

and this is known as the Vlasov equation. Note that $f$ satisfies the Vlasov equation if and only if it is constant along geodesics.

The stress-energy-momentum tensor due to particles of mass $m$ is given by

$$T_{ab}^m(x) = \int_{P_m(x)} f p_a p_b \pi_m \tag{7}$$

and if the Vlasov equation is satisfied then

$$\nabla^a T_{ab}^m = 0 \tag{8}$$

If the gravitational field is due solely to particles of a single mass $m$, represented by $f$, then the coupled Einstein-Vlasov equations, for the metric $g_{ab}$ and the particle distribution $f$ are

$$G_{ab} = 8\pi \int_{P_m} f p_a p_b \pi_m \tag{9}$$

$$\mathcal{L}_g(f) = 0 \tag{10}$$

## 3 Conformally rescaling the massless EV equations

Now we shall restrict to massless, collisionless particles and investigate the transformation properties of the Einstein-Vlasov system under conformal rescaling. For a future-pointing, null co-vector $p_a$, the component $p_0$, either coordinate component or orthonormal-frame component, is determined by

the spatial components $p_i$. Thus we write $\tilde{f}(x^\alpha, p_i)$ for the distribution function for massless particles on $P_0$ in a spacetime $(\tilde{M}, \tilde{g}_{ab})$. Suppose that $\tilde{g}_{ab}$ and $\tilde{f}$ satisfy tilded versions of (9)-(10). Let $g_{ab}$ be the metric defined by

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{11}$$

Under this transformation, null geodesics of $\tilde{g}_{ab}$ become null geodesics of $g_{ab}$. Thus, since $\tilde{f}$ is constant along the geodesic flow in $(\tilde{M}, \tilde{g}_{ab})$, $f \equiv \tilde{f}$ will be constant along the geodesic flow in $(M, g_{ab})$. In fact one calculates that

$$\mathcal{L}_{\tilde{g}} \tilde{f} = \Omega^{-2} \mathcal{L}_g f \tag{12}$$

and it follows that $f$ satisfies the Vlasov equation in $M$.
If we now define

$$T_{ab} = \int f p_a p_b \frac{(-g)_x^{-1/2}}{p^0} d^3 p \tag{13}$$

then the conservation equation $\nabla^a T_{ab} = 0$ holds in $M$, just as the tilded version holds in the physical spacetime. This follows either from the unphysical Vlasov equation, or the relation

$$\tilde{T}_{ab} = \frac{1}{\Omega^2} T_{ab} \tag{14}$$

together with $\tilde{T}_a^a = 0$, which in turn follows from $m = 0$.

With the conformal transformation of the Ricci tensor taken e.g. from ATI, we can now write down the conformal EV equations for $g_{ab}$ and $f$ as

$$R_{ab} = 2\nabla_a \nabla_b \log\Omega - 2\nabla_a \log\Omega \nabla_b \log\Omega$$

$$+ g_{ab}(\Box \log\Omega + 2\nabla_c \log\Omega \nabla^c \log\Omega) + \frac{8\pi}{\Omega^2} \int f p_a p_b \frac{(-g)^{-1/2}}{p^0} d^3 p \tag{15}$$

and

$$\mathcal{L}_g(f) = 0 \tag{16}$$

Note that (15) implies

$$\Box\Omega = \frac{1}{6} R \Omega \tag{17}$$

since ${T_a}^a = 0$.

Now, as $\Omega$ is smooth in $\tilde{M}$ and satisfies the regular, linear wave equation (17) in $M$, one must have that $\Omega$ has a smooth extension onto $M$ (see Racke

1992). It follows that $\Omega$ is a smooth function in $M$.

In the sequel it will be important that $\Omega$ be a good coordinate in $M$, and thus for we make the following assumption:

**Assumption 3.1** *The conformal factor* $\Omega$ *is such that* $\nabla_a \Omega \neq 0$ *at* $\Sigma$.

Thus $\Omega$ corresponds to the time function $Z$ of ATI in the case of a radiation fluid, and for convenience we will henceforth write $\Omega = Z$.

# 4 The initial data: Bondi expansions near $\Sigma$.

The singular nature of the field equation (15) imposes constraints on the data at the singularity $\Sigma$ . To see what these constraints are we shall seek 'Bondi' expansions of the field variables in powers of $Z$ near $\Sigma$ (analogous to the corresponding expansions near future-null-infinity, whence the name). That is, we write

$$h_{ij} = h^0_{ij} + Zh^1_{ij} + \ldots$$

etc, and compare coefficients in (15)-(16). To aid calculation we use coordinates $x^a$ , with $x^0 = Z$ and the $x^i$, $(i = 1, 2, 3)$ taken to be comoving along the integral curves of $\nabla^a Z$. In these coordinates the line element in $M$ can be written

$$ds^2 = \frac{1}{V^2} dZ^2 - h_{ij} dx^i dx^j \tag{18}$$

with $h_{ij}$ positive definite and $V^2 = g^{ab} Z_{,a} Z_{,b}$.

The Vlasov equation for $f = f(x^a, p_i)$ takes the form

$$V^2 p_0 \frac{\partial f}{\partial Z} - h^{ij} p_i \frac{\partial f}{\partial x^j} - \frac{1}{2}\{(\partial_i V^2)(p_0)^2 - (\partial_i h^{mn}) p_m p_n\} \frac{\partial f}{\partial p_i} = 0 \tag{19}$$

where $V^2 (p_0)^2 = h^{ij} p_i p_j$.

The Einstein equations are just

$$Z^2 R_{ab} - 2Z \nabla_a \nabla_b Z + 4 \nabla_a Z \nabla_b Z = g_{ab}(Z \Box Z + V^2) + 8\pi T_{ab} \tag{20}$$

## 4.1 O(1) terms

Considering the lowest order terms in (20) gives

$$8\pi (T_{00})^0 = 3, \quad 8\pi (T_{0i})^0 = 0, \quad 8\pi (T_{ij})^0 = (V^0)^2 h^0_{ij}. \tag{21}$$

The second of these is just an integral constraint on $f^0$, the distribution function at $\Sigma$:

$$\int f^0(x^j, p_k)p_i \; d^3p = 0; \tag{22}$$

the third gives an implicit relation between $f^0$ and $h^0_{ij}$:

$$(V^0)^2 h^0_{ij} = \frac{8\pi}{\sqrt{h^0}} \int \frac{f^0 p_i p_j}{(h^{mn}p_m p_n)^{1/2}} \; d^3p \tag{23}$$

where $h^0 = \det(h^0_{ij})$ and the first is implied by the third since $T_{ab}$ is trace-free.

It is now natural to ask two questions:

1. Given a positive, suitably integrable function $f^0$, does there exist a 3-metric $h^0_{ij}$ satisfying equation (23)?
2. Given a positive-definite 3-metric $h^0_{ij}$ does there exist a positive function $f^0$ satisfying equation (23)?

These questions are answered affirmatively by Theorems 4.1 and 4.2 below.

**Theorem 4.1** Let $f(x^j, p_i)$ be a smooth, positive function on $U \times \mathbb{R}^3$, where $U$ is an open subset of $\mathbb{R}^3$. Suppose that for each $x \in U$

1. $f$ is compactly supported in $p$;
2. $f$ is supported outside some open ball containing $p = 0$;
3. $f$ is not identically zero in $p$.

Then, given a smooth strictly positive function $V(x)$ on $U$, there exists a unique positive definite 3-metric $h_{ij}(x)$ on $U$ satisfying (23). Moreover, the metric $h_{ij}(x)$ is smooth.

*Proof.* The metric is found by a minimisation property. First define the function $F$ by

$$F(b^{11}, b^{22}, b^{33}, b^{12}, b^{13}, b^{23}) = (\det B)^{-1/6} \int (p^T Bp)^{1/2} f \; d^3p \tag{24}$$

where

$$B = \begin{pmatrix} b^{11} & b^{12} & b^{13} \\ b^{12} & b^{22} & b^{23} \\ b^{13} & b^{23} & b^{33} \end{pmatrix} \tag{25}$$

and $B$ is positive definite.

Clearly $F(B) = F(\lambda B)$ for $\lambda \in \mathbb{R}$, so we can think of $F$ as a function defined on a region $S$ in $\mathbb{R}^6$ consisting of points $P = (b^{11}, b^{22}, b^{33}, b^{12}, b^{13}, b^{23})$ for which $B$ is positive definite and $\mathrm{tr}B = 1$.

Now for a couple of Lemmata:

**Lemma 4.1.** The function $F$ has a critical point:

$$\frac{\partial F}{\partial b^{ij}} = 0 \qquad i = 1, 2, 3 \quad j \geq i \tag{26}$$

if and only if the following equation holds

$$a_{ij} \int (p^T Bp)^{1/2} f \ d^3p = 3 \int \frac{f p_i p_j}{(p^T Bp)^{1/2}} \ d^3p \tag{27}$$

where $(a_{ij}) = (b^{ij})^{-1}$.

*Proof.* Note that

$$\frac{\partial(\mathrm{det}B)}{\partial b^{ij}} = \frac{2\hat{b}_{ij}}{2^{(\delta_{ij})}} \tag{28}$$

where $\hat{b}_{ij} = (\mathrm{det}B)a_{ij}$, and there is no summation on the rhs.

Hence

$$\begin{aligned} \frac{\partial F}{\partial b^{ij}} &= -\frac{1}{6}(\mathrm{det}B)^{-1/6}\frac{2}{2^{(\delta_{ij})}}a_{ij} \int (p^T Bp)^{1/2} f \ d^3p \\ &+ (\mathrm{det}B)^{-1/6}\frac{1}{2^{(\delta_{ij})}} \int (p^T Bp)^{-1/2} p_i p_j f \ d^3p \end{aligned} \tag{29}$$

and the lemma follows.

**Lemma 4.2** If $M = (m^{ij})$ is a $3 \times 3$ matrix with $\mathrm{tr}M = 0$, then

$$\delta^2 F \equiv \frac{\partial^2 F}{\partial b^{ij} \partial b^{kl}} m^{ij} m^{kl} 2^{(\delta_{kl}-1)} 2^{(\delta_{ij}-1)} > 0 \tag{30}$$

at any critical point of $F$.

*Proof.* By definition one has

$$\hat{b}_{ij} = \frac{1}{2}\epsilon_{ipq}\epsilon_{jrs}b^{pr}b^{qs} \tag{31}$$

which implies

$$\frac{\partial \hat{b}_{ij}}{\partial b^{kl}} = \sum_{p,r} \frac{1}{2^{(\delta_{kl})}}(\epsilon_{ipk}\epsilon_{jrl} + \epsilon_{ipl}\epsilon_{jrk})b^{pr} \tag{32}$$

and thus

$$\frac{\partial^2 F}{\partial b^{ij}\partial b^{kl}} = 2(1-\delta^{ij})\Bigg\{-\frac{1}{6}\hat{b}_{ij}(\mathrm{det}B)\frac{\partial F}{\partial b^{kl}} - \frac{1}{4}(\mathrm{det}B)^{-1/6}2^{(1-\delta_{kl})}\int \frac{fp_ip_jp_kp_l}{(p^TBp)^{3/2}}\, d^3p$$

$$-\frac{1}{12}(\mathrm{det}B)^{-7/6}2^{(1-\delta_{kl})}\hat{b}_{kl}\int \frac{fp_ip_j}{(p^TBp)^{1/2}}\, d^3p$$

$$-\frac{1}{6}F(B)\left(-((\mathrm{det}B)^{-2})2^{(1-\delta_{kl})}\hat{b}_{kl}\hat{b}_{ij} + (\mathrm{det}B)^{-1}\frac{1}{2^{(\delta_{kl})}}(\epsilon_{ipk}\epsilon_{jrl} + \epsilon_{ipl}\epsilon_{jrk})(b^{pr})\right)\Bigg\} \tag{33}$$

At a critical point we may, without loss of generality, take $B = I_3$ , the $3 \times 3$ identity matrix, for if $b^{ij}$ solves (27) then $I_3$ solves the same equation with $f$ replaced by $\hat{f}$, where $\hat{f}(x,p) = f(x, Lp)$ for some linear transformation $L$, and we can work with $\hat{f}$.

Now drop the hat and evaluate (33) at a critical point to give

$$\frac{\partial^2 F}{\partial b^{ij}b^{kl}} = 2^{(1-\delta_{ij})}2^{(1-\delta_{kl})}\Bigg\{-\frac{1}{4}\int \frac{fp_ip_jp_kp_l}{(p^Tp)^{3/2}}\, d^3p - \frac{1}{12}\delta_{kl}\int \frac{fp_ip_j}{(p^Tp)^{1/2}}\, d^3p$$

$$-\frac{1}{6}F(I_3)\Big(-\delta_{kl}\delta_{ij} + \frac{1}{2}(\epsilon_{ipk}\epsilon_{jrl} + \epsilon_{ipl}\epsilon_{jrk})\delta^{pr}\Big)\Bigg\} \tag{34}$$

so that

$$\frac{\partial^2 F}{\partial b^{ij}\partial b^{kl}}m^{ij}m^{kl}2^{(\delta_{kl}-1)}2^{(\delta_{ij}-1)} = -\frac{1}{4}\int \frac{(p^TMp)^2 f}{(p^Tp)^{3/2}}\, d^3p - \frac{1}{12}\mathrm{tr}M\int \frac{(p^TMp)f}{(p^Tp)^{1/2}}\, d^3p$$

$$-\frac{1}{6}F(I_3)(-(\mathrm{tr}M)^2 + 2\mathrm{tr}\hat{M}) \tag{35}$$

Note that for any $3 \times 3$ matrix $M$ there is the following relation:

$$\mathrm{tr}\hat{M} = \frac{1}{2}\{(\mathrm{tr}M)^2 - \mathrm{tr}M^2\} \tag{36}$$

where $\hat{M}$ is the matrix of cofactors. Hence (35) becomes

$$\delta^2 F = -\frac{1}{4}\int \frac{(p^T M p)^2 f}{(p^T p)^{3/2}}\ d^3p\ + \frac{1}{6}F(I_3)\mathrm{tr}M^2 \tag{37}$$

Now let the eigenvalues of $M$ be $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1 \geq \lambda_2 > 0$, and $\lambda_3 = -(\lambda_1 + \lambda_2)$ (replacing $M$ by $-M$ if necessary).
It follows that

$$|p^T M p|^2 \leq |\lambda_3|^2 |p^T p|^2 \tag{38}$$

and equality holds for only one value of $p$ (up to scale).
Hence

$$\delta^2 F > \frac{1}{12}F(I_3)(\lambda_1 - \lambda_2)^2 \geq 0 \tag{39}$$

Note that we have actually shown that $\hat{F}$, defined by

$$\hat{F}(B) = (\mathrm{det}B)^{-1/6}\int (p^T B p)^{1/2}\hat{f}\ d^3p$$

is such that $\delta^2 \hat{F} > 0$ at $B = I_3$. But we have

$$F(B) \equiv |\mathrm{det}L|^{-4/3}\hat{F}((L^T)^{-1}BL^{-1})$$

It follows that $\delta^2 F > 0$ at a critical point $B$. Hence the lemma.

Suppose next that (27) holds and put $h_{ij} = \lambda a_{ij}$ where

$$\lambda^2 = \left(\frac{8\pi}{3V^2(\mathrm{det}A)^{1/2}}\right)\int (p^T B p)^{1/2} f\ d^3p$$

Then $h_{ij}$ solves (23) and so there is a one-one relation between critical points of $F$ and solutions of (23).
Returning to the set $S$ on which $F$ is defined, we see easily that it is both convex and open in $O \equiv \mathbb{R}^6 \cap \{B :\ \mathrm{tr}B = 1\}$. Since $B$ is +ve definite we get from (36) that $(\mathrm{tr}B)^2 \geq \mathrm{tr}B^2$ and consequently $\max\{|b^{12}|, |b^{13}|, |b^{23}|\} \leq \frac{1}{2}$, so that $S$ is bounded.
Since $S$ is convex it must just consist of radii from some point in $S$, in the directions spanned by the $b^{12}, b^{13}, b^{23}$ axes, and in the plane given by

$$b^{11} + b^{22} + b^{33} = 1$$

The convexity also implies that $S$ has continuous radius, and thus is homeomorphic to an open ball in $\mathbb{R}^5$. In particular $S$ is contractible: it has the

homotopy type of a point. By Lemma 3.2 $\delta^2 F > 0$ at a critical point, when variations are restricted to directions in $S$, so that any critical point of $F$ is a minimum.

Now $F \to \infty$ on the boundary $\partial S$ of $S$ (thinking of $S$ as a subset of $O$), so that $\frac{1}{F}$ is continuous on $\bar{S}$, with $\frac{1}{F} = 0$ on $\partial S$. Now choose $a > 0$ and consider the following subset $Q$ of $S$:

$$Q = \left\{ s \in S : \frac{1}{F} \geq \frac{1}{a} \right\}$$

Since $S$ is bounded and $\frac{1}{F}$ is continuous, one must have that $Q$ is a closed, bounded subset of $O$, disjoint from $\partial S$. Writing $G = -F$ and $b = -a$ now gives that $G^{-1}[b, \infty)$ is compact. Since the critical points of $G$ are all maxima, it follows from theorem 3.5 of (Milnor 1963) that $S$ has the homotopy type of a discrete set of points, one for each critical point of $G$. But we already know that $S$ has the homotopy type of a single point and thus $F$ has just one critical point, as required.

It remains to show that the metric determined by (23) is smooth. To do this first write

$$H_{ij} \equiv \frac{\partial F}{\partial b^{ij}} \tag{40}$$

and suppose that $B^0$ is the critical point of $F$ at $x^0$, so that $H_{ij}(B^0, x^0) = 0$. Now, by hypothesis, $H_{ij}$ is smooth in $B$ and $x$ in a neighbourhood of $(B^0, x^0)$, and we have

$$\frac{\partial H_{ij}}{\partial b^{kl}} = \frac{\partial^2 F}{\partial b^{kl} \partial b^{ij}} \tag{41}$$

But from Lemma 4.2 we have that $\delta^2 F > 0$ at $(B^0, x^0)$, and thus $(d_B H)$ is invertible at this point. From the inverse function theorem it now follows that $B(x)$ defined by $H_{ij}(B, x) = 0$ depends smoothly on $x$.

**Theorem 4.2** Let $h_{ij}(x)$ be a smooth, positive-definite metric on an open set $U \subset \mathbb{R}^3$, with $V$ a smooth, strictly positive function on $U$. Then there exists a smooth positive function $f$ on $U \times \mathbb{R}^3$ satisfying (23), and in fact there are many such $f$ for each $h_{ij}$.

*Proof.* First note that equation (23) can be written in matrix form as

$$(V^2 \sqrt{\det H}/8\pi) H(x) = \int \frac{f(x,p) p p^T}{(p^T H^{-1} p)^{1/2}} \, d^3 p \tag{42}$$

where $H = (h_{ij})$.

Now let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function and $r_1$, $r_2$ positive constants, chosen so that

1. $\phi(x) = 0$ for $|x| \leq r_1$
2. $\phi(x) = 0$ for $|x| \geq r_2$
3. $\phi(x) > 0$ for $r_1 < |x| < r_2$
4.

$$I_3 = \int \frac{\phi(p^T p)}{(p^T p)^{1/2}} p p^T \; d^3p$$

If we let $f(x,p) = V^2(x)\phi(p^T H p)$, then $f$ is smooth and solves (42). Clearly there is considerable freedom in the choice of $\phi$.

### 4.2 O(Z) terms

Let $K_{ab}$ be the second fundamental form in $M$, so that $K_{ij} = -\frac{1}{2}V\partial_z h_{ij}$. A calculation of the O(Z) terms in the space-space components of (20) gives

$$\left(2\delta^i{}_k\delta^j{}_l - \chi^{ij}{}_{kl}\right)(K^0)^{kl} = D^m(\chi^{ij}{}_m) + \frac{1}{2}(K^0)(h^0)^{ij} \qquad (43)$$

where $K^0 = (h^0)^{ij}K^0_{ij}$

$$\chi_{ijkl} \equiv \frac{4\pi}{(V^0)^2\sqrt{h^0}} \int \frac{f^0 p_i p_j p_k p_l}{((h^0)^{mn}p_m p_n)^{3/2}} \; d^3p \qquad (44)$$

$$\chi_{ijk} \equiv -\frac{4\pi}{(V^0)^2\sqrt{h^0}} \int \frac{f^0 p_i p_j p_k}{((h^0)^{mn}p_m p_n)} \; d^3p \qquad (45)$$

and $D_i$ is the covariant derivative operator associated with $(h^0)_{ij}$.
Clearly the eigenvalues of $\chi_{ijkl}$ regarded as a linear transformation on $3 \times 3$ symmetric matrices are positive. Also, from (23), the trace of this linear transformation is 3/2, so that each eigen-value is no greater than 3/2. Thus the 'matrix' acting on $(K^0)_{ij}$ in (43) is invertible, so that the trace-free part of $(K^0)_{ij}$ is determined by $f^0$ through this equation. The trace is not determined but we shall see in section 5 that this can be removed by fixing the conformal gauge.

The O(Z) terms in the space-time and time-time components of (20) impose no more constraints on the data at $Z = 0$.

# 5 Conformal gauge fixing in EV

Inherent in the definition of an isotropic singularity is the freedom to choose a new conformal factor $\bar{\Omega}$ and conformal metric $\bar{g}_{ab}$ by rescaling with a regular non-zero function $\Theta$ according to $\Omega \to \bar{\Omega} = \Theta\Omega$. We now use this freedom to make a conformal gauge choice which should be useful for solving the conformal EV equations in full generality.

First recall the definition of $V$ as $V^2 = g^{ab}\nabla_a Z \nabla_b Z$, and suppose one has

$$\tilde{g}_{ab} = Z^2 g_{ab} = \bar{Z}^2 \bar{g}_{ab} \tag{46}$$

Write $h = \bar{Z}/Z$. Then at $\Sigma$ one has

$$\bar{V}^2 = h^4 V^2 \tag{47}$$

Since $\nabla_a Z \neq 0$ at $\Sigma$, it follows that $h$ can be chosen so that $\bar{V} = 1$ there.

Now write $\hat{g}_{ab} = h^2 \bar{g}_{ab}$, for some new function $h$, with $h = 1$ at $\Sigma$. If $\bar{K}$ is the trace of the second fundamental form of $\bar{g}_{ab}$, with $\hat{K}$ defined similarly, then at $\Sigma$ one gets

$$\hat{K} = \frac{1}{h}(\bar{K} + 3\bar{N}^d \nabla_d \log h) \tag{48}$$

where $\bar{N}^a$ is the unit normal to $\Sigma$ with respect to $\bar{g}_{ab}$. We can get $\hat{K} = 0$ at $\Sigma$ by choosing $h$ so that

$$\bar{N}^d \nabla_d h = -\frac{1}{3}\bar{K} \tag{49}$$

there.

Consider now the following wave equation for $h$:

$$\bar{\Box} h = \frac{1}{6}\bar{R} h \tag{50}$$

with initial data $h = 1,\ \bar{N}^d \nabla_d h = -\frac{1}{3}\bar{K}$.

It is standard (Racke 1992) that this equation has a unique smooth solution $h$ with the given data. If we let $\hat{Z} = \bar{Z}/h$, then $\tilde{g}_{ab} = \hat{Z}^2 \hat{g}_{ab}$, and from (17) there follows

$$\widehat{\Box}\widehat{Z} \equiv 0 \tag{51}$$

We can now prove the following:

**Lemma 5.1** If $(\tilde{M}, \tilde{g}_{ab})$ is a solution of the massless Einstein-Vlasov equations, with an isotropic singularity, and Assumption 3.1 holds, then the conformal factor $Z$ may be chosen so that $V(0) = 1$, $K(0) = 0$, and $Z$ is a harmonic function in $M$. Moreover, these three choices fix the conformal factor completely.

*Proof.* Suppose $\tilde{g}_{ab} = Z^2 g_{ab} = \bar{Z}^2 \bar{g}_{ab}$ , with $Z$, $\bar{Z}$ chosen as above so that

$$V(0) = \bar{V}(0) = 1 \ , \quad K(0) = \bar{K}(0) = 0, \quad \Box Z = \bar{\Box}\bar{Z} \equiv 0$$

Define $h$ by $h = \bar{Z}/Z$, so that $h(0) = 1$ and $(N^d \nabla_d h)(0) = 0$. One has

$$\bar{R} = h^{-2}(R - \frac{6}{h}(\Box h)) \tag{52}$$

But from (17) $R = \bar{R} = 0$, and thus

$$\Box h = 0 \tag{53}$$

Hence $h \equiv 1$, and $\bar{Z} = Z$.

The initial data set for the conformal EV equations at $Z = 0$, in the gauge of Lemma 5.1, can now be summarised as follows:

- The initial distribution function $f^0$ is free data, subject only to the integral constraint (22).
- The initial 3-metric $h^0_{ij}$ is determined by $f^0$ via (23) with $V^0 \equiv 1$.
- The initial second fundamental form $K^0_{ij}$ is determined by $f^0$ via (43) with $V^0 \equiv 1$, $\mathrm{tr}K^0 \equiv 0$

# 6 The Cauchy problem

We have seen that the Einstein-Vlasov system has good behaviour under conformal rescaling and that it is possible to specify Cauchy data for the rescaled equations at an isotropic singularity. In this section we show that if attention is restricted to spacetimes with Bianchi type spatial symmetry, then this Cauchy data determines a unique solution of the field equations.

In this section only we will take the spacetime metric $\tilde{g}_{ab}$ to be of signature $(- + ++)$.

## 6.1 Bianchi type spacetimes

The Bianchi type spacetimes are a class of spatially homogeneous cosmological models having a 3-parameter Lie group $G$ of isometries, transitive on space-like 3-surfaces, the surfaces of homogeneity. They are classified into nine types according to the nature of the Lie algebra associated with $G$ (see Wald 1984 for details). In such spacetimes there exists a cosmic time function $t$ and one-form fields $(e^i)_a$ such that the metric can be written as

$$\tilde{g}_{ab} = -\nabla_a t \nabla_b t + \tilde{a}_{ij}(t)(e^i)_a(e^j)_b \tag{54}$$

The $(e^i)_a$ are left-invariant one-forms, preserved under the action of $G$, and the Bianchi time $t$ is proper-time on the congruence orthogonal to the surfaces of homogeneity.

When field equations are imposed on a spatially-homogeneous spacetime, they typically reduce to a system of ordinary differential equations for the spatial metric $\tilde{a}_{ij}(t)$, together with some matter equations.

## 6.2 Conformal gauge fixing for Bianchi types

Clearly we may assume that the conformal factor $Z$ for Bianchi types is a function only of the cosmic time $t$. There is a natural choice of this function, different from that made in Lemma 5.1, given by the following lemma.

**Lemma 6.1** If it is assumed that the conformal factor satisfies $\nabla_a Z \neq 0$ at $t = 0$, then we may take $Z = t^{1/2}$, where $t$ is the Bianchi time.

*Proof.* Suppose $\tilde{g}_{ab} = Z^2 g_{ab}$, with $g_{ab}$ smooth and $\tilde{g}_{ab}$ a Bianchi type solution of the EV system. Let $V^2 = -g^{ab}\nabla_a Z \nabla_b Z$, with $V = 1$ at $t = 0$ without loss of generality. Define the positive function $g$ by

$$g^2 = \frac{1}{Z^2}\int_0^Z \frac{s}{V(s)}\, ds \tag{55}$$

Now let $\bar{g}_{ab} = g^{-2} g_{ab}$ , $\bar{Z} = gZ$. One then gets $\bar{V} = 1/2$, $\bar{Z} = t^{1/2}$, and it remains to show that $g$ is smooth.

One must have $V^{-1}(s) = 1 + h(s)$ for some smooth $h$ with $h(0) = 0$. Hence $h(s) = sr(s)$ for some smooth $r$, and

$$g^2 = \frac{1}{2} + \frac{1}{Z^2}\int_0^Z s^2 r(s)\, ds$$

It follows that $g$ is continuous. Now $r(s) = r(0) + sp(s)$ with $p$ smooth, and hence

$$g^2 = \frac{1}{2} + \frac{1}{3}r(0)Z + \frac{1}{Z^2}\int_0^Z s^3 p(s)\ ds$$

It now follows that $g$ is $C^1$, and continuing in this way, one shows that $g$ is smooth.

### 6.3 Evolution equations in M

We now wish to write the conformal Einstein-Vlasov equations as an evolution system for the unphysical metric $g_{ab}$ and the particle distribution function $f$ on a manifold $M = [0,T]\times G$. We assume that $f = f(Z,\ v_i)$ , where the $v_i$ are components of 4-momentum in the invariant frame $(e^i)_a$. Note that the metric $g_{ab}$ in $M$ takes the form

$$g_{ab} = -4\nabla_a Z \nabla_b Z + a_{ij}(Z)(e^i)_a(e^j)_b \tag{56}$$

with $a_{ij}$ positive definite.
The Vlasov equation can be written

$$\frac{\partial f}{\partial Z} = \partial_Z f = 2(b^{rs}v_r v_s)^{-1/2} b^{ln} v_k v_l C^k_{jn} \frac{\partial f}{\partial v_j} \tag{57}$$

where $(b^{ij}) = (a_{ij})^{-1}$ and $C^i_{jk}$ are the structure constants of the Lie group G. The spatial components of the Einstein equations, in the basis $(e^i)_a$ are

$$\partial_Z(a_{ij}) = k_{ij} \tag{58}$$

$$\partial_Z(b^{ij}) = -b^{in}b^{jm}k_{mn} \tag{59}$$

$$\begin{aligned}\partial_Z(k_{ij}) = {} & 4C^k_{ck}(C^r_{tj}a_{ir} + a_{jr}C^r_{ti})b^{ct} + 4C^c_{ki}(C^k_{cj} + a_{cm}b^{kt}C^m_{tj}) \\ & +2C^m_{sk}C^r_{tc}a_{jm}a_{ir}b^{kt}b^{cs} + \frac{2}{Z^2}\left\{\frac{32\pi}{(\det\ a)^{1/2}}\int \frac{f v_i v_j}{(v^T b v)^{1/2}}\ d^3v\ - a_{ij}\right\} \\ & -\frac{2}{Z}k_{ij} - \left(\frac{1}{Z}a_{ij} + \frac{1}{2}k_{ij}\right)(b^{rs}k_{rs}) + b^{lq}k_{il}k_{qj}\end{aligned} \tag{60}$$

We wish to remove the second-order pole in $Z$ in (60), so we introduce a new independent variable $Z_{ij}$ according to

$$Z_{ij} = \frac{1}{Z}\left\{\frac{32\pi}{(\det\ a)^{1/2}}\int \frac{f v_i v_j}{(v^T b v)^{1/2}}\ d^3v -\ a_{ij}\right\} \tag{61}$$

Then (60) becomes

$$\partial_Z(k_{ij}) = 4C^c_{kc}(C^r_{tj}a_{ir} + a_{jr}C^r_{ti})b^{kt} + 4C^c_{ki}(C^k_{cj} + a_{cm}b^{kt}C^m_{tj})$$

$$+2C^m_{sk}C^r_{tc}a_{jm}a_{ir}b^{kt}b^{cs} + \frac{1}{Z}(2Z_{ij} - 2k_{ij} - a_{ij}(b^{rs}k_{rs})) - \frac{1}{2}k_{ij}(b^{rs}k_{rs}) + b^{lq}k_{il}k_{qj} \tag{62}$$

and using (57) we get an evolution equation for $Z_{ij}$ :

$$\partial_Z(Z_{ij}) = \frac{1}{Z}\Bigg\{ - Z_{ij} - k_{ij} - \frac{64\pi}{(\det\, a)^{1/2}}C^k_{np}b^{ln}\int \frac{\partial f}{\partial v_p}\frac{v_iv_jv_kv_l}{(v^Tbv)}\; d^3v$$

$$+ \frac{16\pi}{(\det\, a)^{1/2}}(b^{en}b^{fm}k_{mn} - b^{ef}b^{rs}k_{rs})\int \frac{f\;\, v_iv_jv_ev_f}{(v^Tbv)^{3/2}}\; d^3v \Bigg\} \tag{63}$$

Note that if (57)-(59), (62)-(63) are solved, and we define $\bar{Z}_{ij}$ by the rhs of (61) then one gets

$$\partial_Z(\bar{Z}_{ij} - Z_{ij}) = -\frac{1}{Z}(\bar{Z}_{ij} - Z_{ij})$$

and hence $\bar{Z}_{ij} = Z_{ij}$ , and the definition (61) is recovered. Also, from (58)-(59) there follows

$$\partial_Z(a_{ij}b^{jk} - \delta^k_i) = -(a_{ij}b^{js} - \delta^s_i)k_{sn}b^{kn}$$

so that, by a Gronwall estimate, if $a_{ij}b^{jk} = \delta^k_i$ at $Z = 0$ , then $a_{ij}b^{jk} = \delta^k_i$ for all $Z$.

### 6.4 Constraints and initial data

The Hamiltonian constraint is

$$\frac{16\pi}{(\det\, a)^{1/2}}\int f(b^{rs}v_rv_s)^{1/2}\; d^3v = \frac{3}{2} + {}^{(3)}R\; Z^2 - \left(\frac{Z^2}{16}\right) b^{ip}b^{jq}k_{pq}k_{ij}$$

$$+ \left(\frac{Z}{2}\right) b^{ip}b^{jq}k_{pq}a_{ij} + \frac{Z^2}{16}(b^{rs}k_{rs})^2 \tag{64}$$

where ${}^{(3)}R$ is the Ricci scalar of $a_{ij}$.
The momentum constraint is

$$\frac{32\pi}{(\det\, a)^{1/2}}\int fv_i\; d^3v = Z^2(b^{pl}k_{lj}C^j_{pi} + b^{sl}k_{li}C^r_{sr}) \tag{65}$$

Suppose that (57)-(59), (62)-(63) are satisfied, and write (64)-(65) as $C = 0$, $C_i = 0$ respectively. Then one calculates that

$$\partial_Z(Z^2(\det\, a)C) = 0 \quad , \quad \partial_Z C_i = -\frac{1}{2}(b^{rs}k_{rs})C_i$$

Hence $C \equiv 0$ and, by a Gronwall estimate, $C_i \equiv 0$ if $C_i = 0$ initially.
To satisfy the momentum constraint at $Z = 0$ one must have

$$\int f^0 v_i \; d^3v = 0 \tag{66}$$

which is the counterpart of (22), and if the evolution equations are to be satisfied, then from (61)

$$a^0_{ij} = \frac{32\pi}{(\det\, a^0)^{1/2}} \int \frac{f^0 v_i v_j}{(v^T b^0 v)^{1/2}} \; d^3v \tag{67}$$

which is the counterpart of (23) (recall here $V^0 = 1/2$). If (67) is satisfied, then by taking the trace we see that the Hamiltonian constraint is also satisfied initially.

To determine $k^0_{ij}$, $Z^0_{ij}$, note that from (62)-(63)

$$Z^0_{ij} - k^0_{ij} - \frac{1}{2}a^0_{ij}((b^0)^{rs}k^0_{rs}) = 0 \tag{68}$$

and

$$Z^0_{ij} + k^0_{ij} + \frac{64\pi}{(\det\, a^0)^{1/2}} C^k_{np}(b^0)^{ln} \int \frac{\partial f^0}{\partial v_p} \frac{v_i v_j v_k v_l}{(v^T b^0 v)} \; d^3v$$
$$+ \frac{16\pi}{(\det\, a^0)^{1/2}}(-(b^0)^{en}(b^0)^{fm}k^0_{mn} + (b^0)^{ef}(b^0)^{rs}k^0_{rs}) \int \frac{f^0 v_i v_j v_e v_f}{(v^T b^0 v)^{3/2}} \; d^3v \;= 0 \tag{69}$$

Now eliminate $Z^0_{ij}$ to get

$$2k^0_{ij} + \operatorname{tr}k^0(a^0_{ij}) - (k^0)^{ef}\chi_{ijef} + \frac{64\pi}{(\det\, a^0)^{1/2}} C^k_{np}(b^0)^{ln} \int \frac{\partial f^0}{\partial v_p} \frac{v_i v_j v_k v_l}{(v^T b^0 v)} \; d^3v = 0 \tag{70}$$

where

$$\chi_{ijkl} = \frac{16\pi}{(\det\, a^0)^{1/2}} \int \frac{f^0 v_i v_j v_k v_l}{(v^T b^0 v)^{3/2}} \; d^3v \quad , \quad \operatorname{tr}k^0 = (b^0)^{rs}k^0_{rs} \tag{71}$$

Taking the trace of (70) gives

$$0 = \frac{9}{2}\mathrm{tr}k^0 + \frac{64\pi}{(\det\ a)^{1/2}} C^k_{np}(b^0)^{ln} \int \frac{\partial f^0}{\partial v_p} v_k v_l\ d^3v \tag{72}$$

and then (66) implies that $\mathrm{tr}k^0 = 0$. Equation (70) now becomes

$$(\chi_{ijef} - 2a^0_{ie}a^0_{jf})(k^0)^{ef} = -\frac{128\pi}{(\det\ a^0)^{1/2}}(b^0)^{ln}C^k_{n(i} \int \frac{f^0}{(v^T b^0 v)} v_{j)} v_k v_l\ d^3v \tag{73}$$

which corresponds to (43). By the eigenvalue properties of $\chi_{ijkl}$ noted in section 4.3, $k^0_{ij}$ is uniquely determined by $f^0$, and $Z^0_{ij} = k^0_{ij}$ by (68).

In summary, the free data at $Z = 0$ consists of just $f^0(v_i)$ satisfying the integral constraint (66), with $a^0_{ij}$, $k^0_{ij} = Z^0_{ij}$ being determined by (67), (73) respectively. Given such data the constraints are preserved by the evolution, and hence the task is just to solve (57)-(59), (62)-(63).

### 6.5 Solving the evolution equations

We will now use Theorem 1 of (Rendall and Schmidt 1991), together with an iteration technique due to (Rendall 1994), to solve the conformal Einstein-Vlasov equations near $Z = 0$. The result, after a series of lemmas, is Theorem 6.1 below.

Choose $f^0 \in C^\infty_0(\mathbb{R}^3)$ , $f^0 \geq 0$, with $f^0$ supported outside some open ball centred on the origin, and let $a^0_{ij}$, $k^0_{ij} = Z^0_{ij}$ be determined as above. Define

$$m_{ij} = k_{ij} - k^0_{ij} \quad , \quad w_{ij} = Z_{ij} - Z^0_{ij}$$

so that $m^0_{ij} = w^0_{ij} = 0$.
The evolution equations can now be written as follows

$$\partial_Z f = 2(b^{rs}v_r v_s)^{-1/2} b^{ln} v_k v_l C^k_{jn} \frac{\partial f}{\partial v_j} \tag{74}$$

$$\partial_Z(a_{ij}) = k^0_{ij} + m_{ij} \tag{75}$$

$$\partial_Z(b^{ij}) = -b^{in}b^{jm}(k^0_{mn} + m_{mn}) \tag{76}$$

$$\partial_Z(m_{ij}) = 4C^c_{kc}(C^r_{tj}a_{ir} + a_{jr}C^r_{ti})b^{kt} + 4C^c_{ki}(C^k_{cj} + a_{cm}b^{kt}C^m_{tj})$$

$$+2C^m_{sk}C^r_{tc}a_{jm}a_{ir}b^{kt}b^{cs} - \frac{1}{2}(k^0_{ij} + m_{ij})b^{mn}(k^0_{mn} + m_{mn}) + b^{lq}(k^0_{il} + m_{il})(k^0_{qj} + m_{qj})$$

$$+ P_{ij}{}^{mn}(k^0_{mn} + m_{mn}) + \frac{1}{Z}\{-a^0_{ij}(b^0)^{mn}m_{mn} + 2w_{ij} - 2m_{ij}\} \tag{77}$$

$$\partial_Z(w_{ij}) = S^{(1)}_{ij} + S^{(2)}_{ij} + \frac{1}{Z}\Bigg\{ - w_{ij} - m_{ij}$$

$$+ \left(\frac{16\pi}{(\det\, a^0)^{1/2}} \int \frac{f^0 v_i v_j v_e v_f}{(v^T b^0 v)^{3/2}}\, d^3v\right)((b^0)^{en}(b^0)^{fm}m_{mn} - (b^0)^{ef}(b^0)^{mn}m_{mn})\Bigg\} \tag{78}$$

where

$$P_{ij}{}^{mn}(t) = \int_0^1 \{a_{ij}b^{pm}b^{qm}(k^0_{pq} + m_{pq}) - b^{mn}(k^0_{ij} + m_{ij})\}(st)\; ds \tag{79}$$

and

$$S^{(1)}_{ij}(t) = \int_0^1 -\frac{64\pi}{(a)^{1/2}} C^k_{np}\Bigg\{(b^{rs}(k^0_{rs}+m_{rs})b^{ln} - b^{lp}b^{nq}(k^0_{pq}+m_{pq}))\int \frac{\partial f}{\partial v_p}\frac{v_i v_j v_k v_l}{(v^T b v)}\, d^3v$$

$$+ b^{ln}b^{ru}b^{sv}(k^0_{uv} + m_{uv})\int \frac{\partial f}{\partial v_p}\frac{v_i v_j v_k v_l v_r v_s}{(v^T b v)^2}\, d^3v$$

$$+ 2C^w_{qr}b^{rs}b^{ln}\int \frac{\partial}{\partial v_p}\left(\frac{v_w v_s}{(v^T b v)^{1/2}}\frac{\partial f}{\partial v_q}\right)\frac{v_i v_j v_k v_l}{(v^T b v)}\, d^3v\Bigg\}(st)\; ds \tag{80}$$

$$S^{(2)}_{ij}(t) = (k^0_{mn} + m_{mn})\int_0^1 \frac{16\pi}{(a)^{1/2}}\Bigg\{2C^k_{pr}b^{lr}\int \frac{\partial f}{\partial v_p}\frac{v_i v_j v_e v_f v_k v_l}{(v^T b v)^2}\, d^3v$$

$$+(k^0_{rs} + m_{rs})\Bigg[\frac{3}{2}b^{pr}b^{qs}(b^{en}b^{fm} - b^{ef}b^{mn})\int \frac{f v_i v_j v_e v_f v_p v_q}{(v^T b v)^{5/2}}\, d^3v$$

$$+\left(\int \frac{f v_i v_j v_e v_f}{(v^T b v)^{3/2}}\, d^3v\right)\Big(b^{rs}(b^{en}b^{fn} - b^{ef}b^{mn})$$

$$- b^{er}b^{ns}b^{fm} - b^{en}b^{fr}b^{ms} + b^{mn}b^{er}b^{fs} + b^{ef}b^{mr}b^{ns}\Big)\Bigg]\Bigg\}(st)\; ds \tag{81}$$

where $a = \det(a_{ij})$.

Suppose now that $f^0$ is supported outside a ball of radius $B_1$ centred on the origin. Let $r = (v^T v)^{1/2}$, and let $\phi(v_i)$ be a smooth function such

that $\phi = 1$ on $r > B_2$, $B_2 < B_1$, $\phi = 0$ on $r < B_3$ , $B_3 < B_2$, and $0 \leq \phi \leq 1$ elsewhere. Replace the Vlasov equation (57) by

$$\frac{\partial f}{\partial Z} = 2\phi(b^{rs}v_r v_s)^{-(1/2)} b^{ln} v_k v_l C^k_{jn} \frac{\partial f}{\partial v_j} \tag{82}$$

Equations (77)-(78) can be written

$$\partial_Z(u) + \frac{1}{Z} N u = G(a, b, u, f) \tag{83}$$

where $u$ stands for $m$, $w$, and $N$ is a constant matrix containing just $a^0_{ij}$, $(b^0)^{ij}$. $N$ has desirable properties with regard to application of the theorem of Rendall and Schmidt (1991), given by the following lemma:

**Lemma 6.2** All the eigenvalues of $N$ have positive real part and $N$ is diagonalisable.

*Proof.* The eigenvalue equation for $N$ is

$$2m_{ij} + m a^0_{ij} - 2w_{ij} = \lambda m_{ij} \tag{84}$$

$$m_{ij} + \frac{1}{2} m a^0_{ij} + w_{ij} - \chi_{ijkl} m^{kl} = \lambda w_{ij} \tag{85}$$

where $m_{ij}$, $w_{ij}$ are components of an eigenvector, $m = (b^0)^{ij} m_{ij}$, and $\lambda$ is the eigen-value.

Hence $w_{ij} = \frac{1}{2}(2 - \lambda) m_{ij} + \frac{1}{2} m a^0_{ij}$, and substituting in (85) gives

$$(\lambda^2 - 3\lambda + 4) m_{ij} - (\lambda - 2) m a^0_{ij} = 2\chi_{ijkl} m^{kl} \tag{86}$$

We know from (67) that $(b^0)^{ij} \chi_{ijkl} = \frac{1}{2} a^0_{kl}$, and so taking the trace of (86) gives

$$(\lambda - 3)^2 m = 0 \tag{87}$$

If $m \neq 0$ then $\lambda = 3$ and an eigenvector is $w_{ij} = m_{ij} = \frac{1}{3} m a^0_{ij}$ . If $\lambda \neq 3$ then $m = 0$ and (86) becomes

$$(\lambda^2 - 3\lambda + 4) m^{ij} = 2 m^{kl} \chi^{ij}{}_{kl} \tag{88}$$

and we must consider eigenvalues of $\chi^{ij}{}_{kl}$, which we may regard as a $9 \times 9$ symmetric matrix, with 9 real eigenvalues and 9 linearly independent eigenvectors. So suppose

$$\chi^{ij}{}_{kl} m^{kl} = \mu m^{ij} \tag{89}$$

Then $m_{ij} = m_{ji}$ and we may work in an invariant basis where $a^0_{ij} = \delta_{ij},\ (m_{ij}) = \mathrm{diag}(m_1,\ m_2,\ m_3)$. Clearly $\mu$ must be positive since

$$\chi_{ijkl} m^{ij} m^{kl} \geq 0$$

It follows that

$$\mu |m_i| = 16\pi \Big| \int \frac{f^0}{(v^T v)^{3/2}} v_i^2 (v_1^2 m_1 + v_2^2 m_2 + v_3^2 m_3)\ d^3 v \Big|$$

$$\leq 16\pi \int \frac{f^0 v_i^2}{(v^T v)^{3/2}} (v_1^2 |m_1| + v_2^2 |m_2| + v_3^2 |m_3|)\ d^3 v$$

and hence

$$\mu \sum_{i=1}^{3} |m_i| \leq 16\pi \int \frac{f^0}{(v^T v)^{1/2}} (v_1^2 |m_1| + v_2^2 |m_2| + v_3^2 |m_3|)\ d^3 v \ = \frac{1}{2} \sum_{i=1}^{3} |m_i|$$

by (67), so that $\mu \leq \frac{1}{2}$.
Going back to (88) one gets that $\lambda = \frac{1}{2}(3 \pm [8\mu - 7]^{1/2})$ with $0 \leq \mu \leq \frac{1}{2}$. Thus $\lambda$ has positive real part, and there exist two $\lambda' s$ for each of the nine linearly independent $m_{ij}$ . One has $w_{ij} = \frac{1}{2}(2 - \lambda) m_{ij}$, and this gives 18 linearly independent eigenvectors of $N$.

Lemma 6.2 gives that there exists an 18×18 matrix $L$ such that equations (83) may be written

$$\partial_Z (y_\alpha) + \frac{1}{Z} \lambda_\alpha y_\alpha = \ H_\alpha(a, b, y, f) \tag{90}$$

where $y = Lu,\ H(a, b, y, f) = LG(a, b, L^{-1} y, f)$ , and $\lambda_\alpha$ are the eigenvalues of $N$.
The equations to be solved are now (75), (76), (82), (90).

Consider first the modified Vlasov equation (82). The characteristics of this equation are defined as the solutions $V_j(s, t, v)$ of the system

$$\frac{dV_j}{ds} = -2\phi(V)(b^{rs} V_r V_s)^{-1/2} b^{ln} V_k V_l C^k_{jn} \tag{91}$$

with $V_j(t, t, v) = v_j$. Then the solution of (82) is given by $f(t, v_i) = f^0(V_j(0, t, v_i))$ , and is smooth by standard theory.

Now define an iteration as follows: Let $f^0,\ a^0_{ij},\ (b^0)^{ij}$ be the initial data as above, and let $m^0_{ij} = w^0_{ij} = 0$. If smooth iterates $a^n,\ b^n,\ m^n,\ w^n,\ f^n$ are given for some $n$, let $y^n = Lu^n$ , and determine $V^{n+1}$ by solving (91) with $b^n$ in the rhs, and let $f^{n+1}(t,v) = f^0(V^{n+1}(0,t,v))$ which solves the $n$-th Vlasov equation. Now substitute $a^n,\ b^n,\ y^n,\ f^{n+1}$ into the rhs of (75), (76), (90). By Theorem 1 of (Rendall and Schmidt 1991) these linear ODE's have a unique smooth solution $a^{n+1},\ b^{n+1},\ y^{n+1}$ on some small time interval $[0, T'_{n+1})$. Let $[0, T_{n+1})$ be the largest time interval on which (det $a^n$) remains strictly positive. Then by standard theory for regular, linear ODE's, the (n+1)th solution can be extended uniquely and smoothly onto $[0, T_{n+1})$.

Let $|a|$ be the maximum modulus of any component of $(a_{ij})$ , with a similar definition for other quantities, and suppose that $\forall n < N$ the following bounds hold:

$$|a^n - a^0| \le A_1 \qquad |b^n - b^0| \le A_2$$

$$|m^n| \le A_3 \qquad |w^n| \le A_4$$

$$(\det\ b^n)^{-1} < A_6 \ \ (A_6 > \ (\det\ b^0)^{-1}) \tag{92}$$

Suppose also that $f^n(t,v) \neq 0 \ \Rightarrow \ (v^T v)^{1/2} \le A_5$

Now the speed of propagation for the nth Vlasov equation, at the point $v_i$ will be bounded by $C_1 (v^T v)^{1/2}$ , with $C_1 = C_1(A_2, A_6)$. Hence if $R = \text{diam supp } f^N(t)$ , then

$$\frac{dR}{dt} \le C_1 R \quad \Rightarrow \quad R \le \ P_0 \exp(C_1 t) \tag{93}$$

where $P_0 =$ diam supp $f^0$.

Now (75), (76) give

$$|a^N - a^0| \le C_2 t \qquad |b^N - b^0| \le C_3 t \tag{94}$$

and then one gets

$$(\det\ b^N)^{-1} < A_6 \tag{95}$$

for $t$ in a small enough interval $[0, T]$, where $T = T(A_i)$. Since $f^0$ is bounded one also gets: $|H^n| \le C_4(A_i)$ for $n < N$.

From (90) there follows

$$y^N_\alpha(t) = t^{-\lambda_\alpha} \int_0^t s^{\lambda_\alpha} H^{N-1}_\alpha(s)\ ds \tag{96}$$

and hence $|y_\alpha^N(t)| \leq C_4 t$ . Also, from the bound on the speed of propagation, if $f^0$ is supported outside $r = B_1$ , then $f^N$ is supported outside $r = B_1 \exp(-C_1 t)$.
Now choose $A_5 > P_0$ , and choose $T = T(A_i, C_j)$ small enough so that $R < A_5$, and for $t \in [0,T]$ $f^N(t)$ is supported on $\{v : \phi(v) = 1\}$, and the following bounds hold:

$$|a^N - a^0| \leq A_1 \qquad |b^N - b^0| \leq A_2$$

$$|m^N| \leq A_3 \qquad |w^N| \leq A_4 \tag{97}$$

Then by induction all iterates exist, are smooth, and are uniformly bounded on $[0,T]$. Also, $\forall n$, $f^n$ is supported on $\{v : \phi(v) = 1\}$, and diam supp$(f^n) \leq A_5$.
For the difference between successive iterates one has the following estimates:

$$|a^{n+1} - a^n|(t) \leq C \int_0^t \ |y^n - y^{n-1}|(s)\ ds \tag{98}$$

$$|b^{n+1} - b^n|(t) \leq C \int_0^t \ |b^n - b^{n-1}|(s) + |y^n - y^{n-1}|(s)\ ds \tag{99}$$

$$|y^{n+1} - y^n|(t) \leq C \int_0^t \Big\{ (|a^n - a^{n-1}| + |b^n - b^{n-1}| + |y^n - y^{n-1}|)(s)$$

$$+ \Big( \int_0^1 \ (|a^n - a^{n-1}| + |b^n - b^{n-1}| + |y^n - y^{n-1}| + \|f^{n+1} - f^n\|_\infty)(sp)\ dp \Big) \Big\}\ ds \tag{100}$$

Equation (98) comes from (75), (99) comes from (76), and (100) comes from (96).

From the characteristic equation (91) one gets an inequality similar to that in (93) and hence the $V^n$ are uniformly bounded, above and below for $v$ in a compact set with $\phi(v) = 1$ , and $t \leq T$, $0 \leq s \leq t$. Now suppose $v \in \{v : \phi(v) = 1\} \cap \{v : (v^T v)^{1/2} < A_5\}$. Then from (91) there follows the estimate:

$$\Big| \frac{dV^{n+1}}{ds} - \frac{dV^n}{ds} \Big| (s,t,v) \leq C\{|V^{n+1} - V^n| + |b^n - b^{n-1}|\} \tag{101}$$

for $t \leq T$, $0 \leq s \leq t$.
Define the quantity $\alpha^n(t)$ by

$$\alpha^n(t) = \ \sup\{|V^{n+1} - V^n|(s,t,v) : \ 0 \leq s \leq t,\ r < A_5,\ \phi(v) = 1\}$$

$$+ |a^{n+1} - a^n|(t) + |b^{n+1} - b^n|(t) + |y^{n+1} - y^n|(t) \tag{102}$$

Then

$$\|f^{n+1} - f^n\|_\infty(t) \leq \ \|f^0\|_{C^1}(\alpha^n(t)) \tag{103}$$

Combining the above inequalities gives

$$\alpha^n(t) \leq \ C\int_0^t \Big\{\alpha^n(s) + \alpha^{n-1}(s) + \int_0^1 (\alpha(sp) + \alpha^{n-1}(sp))\ dp\Big\}\ ds \tag{104}$$

Now Gronwall's inequality implies

$$\alpha^n(t) \leq \ C\int_0^t \Big\{\alpha^{n-1}(s) + \Big(\int_0^1 \alpha^n(sp) + \alpha^{n-1}(sp)\ dp\Big)\Big\}\ ds \tag{105}$$

for $t \in \ [0, T]$, and since the rhs is increasing , this implies

$$\sup_{(0\leq s\leq t)}\alpha^n(s) \ \leq C\int_0^t \Big\{\alpha^{n-1}(s) + \Big(\int_0^1 (\alpha^n(sp) + \alpha^{n-1}(sp))\ dp\Big)\Big\}ds$$

$$\leq \ C_1 t\Big(\sup_{(0\leq s\leq t)}\alpha^n(s)\Big) + \ C_2\int_0^t \sup_{(0\leq p\leq s)}(\alpha^{n-1}(p))\ ds \tag{106}$$

Put $\beta^n(t) = \ \sup_{(0\leq s\leq t)}\alpha^n(s)$. Then (106) is just

$$\beta^n(t)(1 - C_1 t) \leq \ C_2\int_0^t \ \beta^{n-1}(s)\ ds \tag{107}$$

Choose $T_1 \leq T$ so that $(1 - C_1 t) \geq \frac{1}{2}$ for $t \in [0, T_1]$. It follows now that

$$\beta^n(t) \leq 2C_2\int_0^t \ \beta^{n-1}(s)\ ds \tag{108}$$

and hence

$$\beta^n(t) \leq \ C^{n-2}\|\beta^2\|_\infty t^{n-2}/(n-2)! \tag{109}$$

Therefore $V^n$, $b^n$, $a^n$, $y^n$ are uniformly Cauchy on $[0, T_1]$. It follows that $a^n$, $b^n$, $y^n$ have continuous limits $a(t)$, $b(t)$, $y(t)$ on $[0, T_1]$, and $V^n(s, t, v)$ has a limit $V(s, t, v)$ continuous in $s, t$. From (75), (76), one gets that $a, b$ are $C^1$, and from (91) $V(s, t, v)$ is $C^1$ in $s, t$, so that $a, b, V$ in fact satisfy these equations. It is then standard (Taylor 1996) that $V$ is $C^1$ in $v$. If we put $f(t, v) = f^0(V(0, t, v))$, then $f$ is supported on $\{v : \phi(v) = 1\}$ and hence satisfies the Vlasov equation (74) .
Note that we can differentiate (91), and so $V$ is $C^2$ along with $f$.

Clearly from (90), (75), (76), (91) we get uniform convergence of the time derivatives of all iterates on $[\epsilon, T_1]$ for arbitrary $0 < \epsilon < T_1$ , and inductively it follows that all the sequential limits are smooth on $(0, T_1]$.

If we let $u = (m, w) = L^{-1}y$ as before, then $u$ is continuous on $[0, T_1]$ , smooth on $(0, T_1]$ , and satisfies (83) there. Putting $k_{ij} = k^0_{ij} + m_{ij},\ Z_{ij} = Z^0_{ij} + w_{ij}$ gives that $k_{ij},\ Z_{ij}$ are continuous on $[0, T_1]$ , smooth on $(0, T_1]$ , and satisfy (62), (63) there. These equations can be written

$$\frac{dx}{dZ} + \frac{1}{Z}N(a, b, f)x = G_1(a, b, x) + \frac{1}{Z}G_2(a, b, f) \tag{110}$$

where $x$ stands for $k_{ij},\ Z_{ij}$ , $G_1$ is quadratic in $x$, $G_2$ accounts for the terms $64\pi C^k_{np} \ldots$, and $Z \in (0, T_1]$.

Now regard $a, b, f$ as known $C^1$ functions on $[0, T_1]$. Then (110) implies

$$Z\frac{dx}{dZ} + Nx = (N - N(Z))x + ZF_1(Z, x) + F_2(Z) \tag{111}$$

where $N$ is as in (83), $F_1$ is $C^1$ in $Z$ and quadratic in $x$, and $F_2$ is $C^1$. We can now differentiate (111) on $(0, T_1]$ to get

$$Z\frac{dq}{dZ} + (I + N)q = ZR(Z)q + S(Z) \tag{112}$$

where $q = \frac{dx}{dZ}$ and $R,\ S$ are continuous functions on $[0, T_1]$.

It follows by a lemma of (Rendall and Schmidt 1991) that (112) has a unique $C^1$ solution on $(0, T_1]$, and that this extends to a continuous solution of the associated integral equation on $[0, T_1]$. Hence $x$ is $C^1$, and $(a,\ b,\ k,\ z,\ f)$ is a classical solution of the conformal Einstein-Vlasov equations. Inductively one gets that this solution is in fact $C^\infty$.

If two solutions with the same initial data are given, then it is possible to derive an inequality for their difference similar to that obtained for the quantity $\beta^n$. It follows that the solution constructed above is unique.

We have therefore proved the following:

**Theorem 6.1.** Let $G$ be a 3-dimensional Lie group of some Bianchi type $n$, and let $(e^i)_a$ be a basis of left-invariant one-form fields on $G$ . Suppose

we are given a smooth function $F^0$ on the cotangent bundle of $G$ such that $F^0(x,p) = f^0(p_i)$ where $p_i$ are the components of $p$ in the frame $(e^i)_a$. Suppose also that $f^0$ is compactly supported, supported outside a neighbourhood of the origin, and the integral constraint (66) holds. Then there exists exactly one smooth Bianchi type $n$ solution $(\tilde{f}(t,p_i), \tilde{g}_{ij}(t))$ of the massless Einstein-Vlasov equations on $G \times (0,T]$ with an isotropic singularity, satisfying $\tilde{f}(t,p_i) \to f^0(p_i)$ as $t \to 0$ .

Note that since the initial 3-metric $a^0_{ij}$ is determined by $f^0$ only through the integral relation (67) there will be many (anisotropic) $f^0$ which give rise to the same $a^0_{ij}$. But then $k^0_{ij}$ is determined by the third and fourth moments of $f^0$, and so we expect in general to be able to get different values of $k^0_{ij}$ , and thus different 4-geometries, from the same starting metric $a^0_{ij}$. This is in contrast to the perfect fluid cosmologies of ATI, where it was shown that the 4-geometry is in 1-1 correspondence with the 3-geometry on $\Sigma$.

### 6.6 FRW data

The Bianchi types I, V and IX contain, respectively, the $k = 0, -1$ and $+1$ FRW models. We now discuss the question of which $f^0$ constitute FRW data for the conformal EV equations in these Bianchi types.

**(a) Type I**
The situation here is very simple, as all the $C^i_{jk}$ are zero, and one solution of the field equations is $\tilde{f}(Z,p_i) = f^0(p_i)$, $\tilde{g}_{ij} = Z^2 a^0_{ij}$ , for any admissible $f^0$. Hence Theorem 6.1 gives that all Bianchi I solutions of massless EV with an isotropic singularity are in fact FRW and the Weyl tensor is always zero.

**(b) Type V**
Here the structure constants are

$$C^i_{jk} = \delta_j{}^i a_k - \delta_k{}^i a_j \tag{113}$$

for some vector $a_i$.
All type V 3-metrics have constant (negative) curvature, and for $f^0$ to be FRW data one must certainly have also $k^0_{ij} = 0$, since $\mathrm{tr}k^0 = 0$. An isotropic $f^0$ satisfies this condition by (73), and gives the FRW evolution $\tilde{f} = f^0$, $\tilde{g}_{ij} = R^2(Z) a^0_{ij}$, for some function $R$. It is not clear whether $f^0$ *must* be isotropic to give an FRW solution (see (117) below).
Leaving this question aside for the moment, we consider a related question,

namely whether one can pick an anisotropic $f^0$ so that the Weyl tensor vanishes at $\Sigma$ , but the evolution given by Theorem 6.1 is not FRW. We calculate that the electric and magnetic parts of the Weyl tensor defined relative to $\nabla^a Z$ vanish at $Z=0$ if and only if

$$D_m k^0_{l(j} e_{i)}{}^{lm} = 0 \tag{114}$$

$$k^0_{ie}(k^0)^e{}_j + P_{ij} + \lambda a^0_{ij} = 0 \tag{115}$$

where $e_{ijk}$ is the volume form, $\lambda$ just takes care of the trace and

$$\begin{aligned}
P_{ij} \equiv \frac{32\pi}{(a^0)^{1/2}} C^s_{tr} \int \frac{\partial f^0}{\partial v_t} \frac{(v^T k^0 v)(b^0)^{qr} v_q v_s v_i v_j}{(v^T b^0 v)^{5/2}}\, d^3v + \frac{24\pi}{(a^0)^{1/2}} \int \frac{f^0 (v^T k^0 v)^2 v_i v_j}{(v^T b^0 v)^{5/2}}\, d^3v \\
-2(k^0)^{en}(k^0)^f{}_n \chi_{ijef} \\
-\frac{64\pi}{(a^0)^{1/2}} C^k_{np} \Bigg[ -(k^0)^{ln} \int \frac{f^0}{(v^T b^0 v)} \left( \{\delta^p{}_i v_j v_k v_l + \text{perm } ijkl\} - \frac{2 v_i v_j v_k v_l v_m (b^0)^{mp}}{(v^T b^0 v)} \right) d^3v \\
+2C^s_{tr}(b^0)^{ln} \int \frac{\partial f^0}{\partial v_t} \frac{v_s v_q (b^0)^{qr}}{(v^T b^0 v)^{3/2}} \{\delta^p{}_i v_j v_k v_l + \text{perm } ijkl\}\, d^3v \\
+(b^0)^{ln} \int \frac{f^0}{(v^T b^0 v)^2} (v^T k^0 v) \{\delta^p{}_i v_j v_k v_l + \text{perm } ijkl\}\, d^3v \\
-2(b^0)^{ln} \int \frac{f^0}{(v^T b^0 v)^2} v_i v_j v_k v_l v_m \left( \frac{2(b^0)^{mp}(v^T k^0 v)}{(v^T b^0 v)} - (k^0)^{mp} \right) d^3v \Bigg]
\end{aligned} \tag{116}$$

where $a^0 = \det a^0_{ij}$.

Now suppose that one has an FRW Einstein-Vlasov metric: $a_{ij} = R^2(Z) a^0_{ij}$. We then calculate from the field equations that the following condition must hold:

$$\int \frac{\partial^n f}{\partial Z^n} \frac{v_i v_j}{(v^T b^0 v)^{1/2}}\, d^3v \Big|_\Sigma = 0. \tag{117}$$

for all $n \geq 1$ (it is not clear whether (117) this is sufficient to force isotropy of $f^)$).

From the Vlasov equation (57) we get for $n = 1,\ 2$ that conditions (117) are respectively:

$$\chi_{ijk} a^k = 0 \tag{118}$$

$$10\chi_{ijkl} a^k a^l - 2a_i a_j - (a_k a^k) a^0_{ij} = 0 \tag{119}$$

Suppose now that we pick an $f^0$ so that (118) and (119) hold, but (117) fails to hold for some $n > 2$ (note that (118) and (119), together with (66) and (67), amount to a set of conditions on the first four moments of $f^0$ only, so that there are many non-isotropic $f^0$ satisfying them, while (117) for larger and larger $n$ imposes conditions on ever larger moments of $f^0$). A calculation then shows that (114) and (115) are satisfied with $k^0_{ij} = 0$, so that the Weyl tensor vanishes at $\Sigma$. Thus it is possible, with such an $f^0$, to generate a non-FRW Einstein-Vlasov cosmology with an initially vanishing Weyl tensor.

**(c) Type IX**

Here there exists an invariant basis for which $C^i_{jk} = \epsilon_{ijk}$, and a 3-metric $h_{ij}$ has constant curvature iff $h_{ij} \propto \delta_{ij}$ in this basis. So if we let $a^0_{ij}$ be such a metric then $k^0_{ij} = 0$ by (73) and the cosmology given by Theorem 6.1 is then (k=+1) FRW with $\tilde{f}(Z, p_i) = f^0(p_i)$, $\tilde{g}_{ij} = R^2(Z)a^0_{ij}$, for some $R(Z)$ regardless of the choice of $f^0$. Thus FRW data in type IX is any admissible $f^0$ which gives rise to a constant curvature 3-metric.
As in type V one can ask whether an $f^0$ can be chosen so that the Weyl tensor vanishes at $\Sigma$, while the 4-geometry departs from FRW shape. Here the condition $C_{abcd}(0) = 0$ amounts to the following

$$k^0_{ij} = 0 \tag{120}$$

$$\left(9\delta_i{}^e\delta_j{}^f - \chi_{ij}{}^{ef}\right) S_{ef} = \frac{5}{8} Q_{ij} \tag{121}$$

where $S_{ij}$ is the trace-free Ricci tensor of $a^0_{ij}$ and

$$Q_{ij} \equiv -16(b^0)^{ln}(b^0)^{rs} C^t_{ps} C^k_{n(i} \{\delta_k{}^p \chi_{rtl|j)} + \delta_l{}^p \chi_{rtk|j)} + \delta_{j)}{}^p \chi_{rtkl}\} \tag{122}$$

We have not been able to find a solution of these equations for which $S_{ij} \neq 0$ and it seems unlikely that there is one. If this is so, then any $f^0$ which makes the initial Weyl tensor vanish is FRW data.

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